MAS381 Crib Sheet

January 3, 2019

1 Complex Functions

1.1 Definition of a harmonic function:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1}$$

1.2 Cauchy-Reimann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

1.3 Check for convergence of a complex series:

$$L = \lim_{n\to\infty} \mid \frac{c_{n+1}}{c_n} \mid \tag{3}$$

- Convergent for $L < 1$
- Divergent for $L > 1$
- Unknown for $L = 1$

1.4 Radius of convergence:

For positive powers:

$$\rho = \frac{1}{L} = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| \tag{4}$$

For negative powers:

$$R = \lim_{n\to\infty} |\frac{a_{-n-1}}{a_{-n}}| \tag{5}$$

1.5 Taylor's theorem:

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + ... + \frac{(z-a)^n}{n!}f^n(a) + ... \quad (6)$$

1.6 Laurent series:

$$\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n, & |z| < 1 \\ -\sum_{n=1}^{\infty} z^{-n}, & |z| > 1 \end{cases} \quad (7)$$

2 Complex Integration

2.1 Cauchy integral formula:

$$\oint_C \frac{f(z)}{z-a} dz = \begin{cases} 2\pi i f(a), & a \in D \\ 0, & a \notin D \end{cases} \quad (8)$$

2.2 Residue theorem:

$$a_{-1} = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z) \quad (9)$$

If z_0 is a simple pole:

$$a_{-1} = \lim_{z \to z_0} (z-z_0) f(z) \quad (10)$$

$$\oint f(z) dz = 2\pi i (a_{-1} + a_{-2} + ... + a_n) \quad (11)$$

3 Vector Calculus

3.1 Gradient of a scalar field:

$$\nabla(f) = (f_x, f_y, f_z) \quad (12)$$

3.2 Divergence operator:

$$\nabla \cdot \boldsymbol{u} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (f, g, h) = f_x + g_y + h_z \quad (13)$$

3.3 Curl operator:

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ f & g & h \end{vmatrix} = (h_y - g_z, f_z - h_x, g_x - f_y) \tag{14}$$

3.4 Laplacian operator:

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \tag{15}$$

4 Integration

4.1 Work done against a force:

$$-\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} \tag{16}$$

4.2 Flux of a vector through a curve:

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{n} \tag{17}$$

Where:

$$d\boldsymbol{n} = (dy, -dx) \tag{18}$$

4.3 Two-dimensional divergence theorem:

$$\iint_D div(\boldsymbol{u})dA = \int_C \boldsymbol{u} \cdot d\boldsymbol{n} \tag{19}$$

4.4 Green's theorem:

$$\iint_D curl(\boldsymbol{u}) \cdot d\boldsymbol{A} = \int_C \boldsymbol{u} \cdot d\boldsymbol{r} \tag{20}$$

4.5 Three-dimensional divergence theorem:

$$\iiint_E div(\boldsymbol{u})dV = \iint_S \boldsymbol{u} \cdot d\boldsymbol{A} \tag{21}$$

4.6 Sokes' theorem:

$$\iint_S curl(\boldsymbol{u}) \cdot d\boldsymbol{A} = \pm \int_C \boldsymbol{u} \cdot d\boldsymbol{r} \tag{22}$$

4.7 Parametrisations:

Circle:

$$\boldsymbol{r} = (rcos(\theta), rsin(\theta)) \tag{23}$$

Hemisphere:

$$\boldsymbol{r} = (rsin(\phi)cos(\theta), rsin(\phi)sin(\theta), rcos(\phi)) \tag{24}$$

Cylinder:

$$\boldsymbol{r} = (1 + cos(s), 1 + sin(s), t) \tag{25}$$

5 Useful Trigonometric Identities

$$cos^2(\theta)sin(\theta) = \frac{1}{4}[sin(3\theta) + sin(\theta)] \tag{26}$$

$$cos(\alpha)cos(\beta) = \frac{1}{2}[cos(\beta + \alpha) + cos(\beta - \alpha)] \tag{27}$$

$$cos^2(\theta) = \frac{1}{2}[1 + cos(2\theta)] \tag{28}$$

$$sin^2(\theta) = \frac{1}{2}[1 - cos(2\theta)] \tag{29}$$